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THE DESIGN OF NONHOMOGENEOUS EQUI-STRENGTH ANNULAR DISCS OF VARIABLE THICKNESS UNDER INTERNAL AND EXTERNAL PRESSURES

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Abstract—The optimal distributions of Young's modulus and thickness which guarantee satisfaction of the Tresca yield criterion simultaneously at all points within the annular discs are presented. The prescribed stress state takes place when the quasistatic pressures upon the contours reach their limit values. An analytical solution has been obtained for the case of an arbitrary distribution of thickness (Young's modulus) by means of a semireciprocal method. Assuming the disc thickness (Young's modulus) to be a power function of radius relations between the limit values of pressures and the optimal distributions of Young's modulus (thickness) have been presented. For the case of prescribed linear distributions of radial stress, formulae for the determination of optimal distributions of thickness and Young's modulus by means of a reciprocal method have been obtained. The regions of variation of pressures in which homogeneous or annular discs of optimal distributions of Young's modulus and thickness respectively will remain elastic, are determined. The advantages of discs of optimal distributions of Young's modulus and thickness compared with homogeneous discs and discs with constant thickness are discussed.

NOTATION

$\sigma_r^0, \sigma_\theta^0$	radial and hoop stress components
u^0	radial displacement
p_1^0	pressure upon inner contour
p_2^0	pressure upon external contour
a	inner radius
b	outer radius
ρ	radial coordinate
σ^0	yield stress
v	Poisson's ratio
$E^0(\rho)$	Young's modulus distribution
$h^0(\rho)$	thickness distribution.

It is convenient to introduce the following dimensionless quantities:

$$r = \frac{\rho}{b};\quad \alpha = \frac{a}{b};\quad \sigma_\theta = \frac{\sigma_\theta^0}{\sigma^0};\quad \sigma_r = \frac{\sigma_r^0}{\sigma^0};\quad h(r) = \frac{h^0(\rho)}{h^0(b)};$$

$$p_1 = \frac{p_1^0}{\sigma^0};\quad p_2 = \frac{p_2^0}{\sigma^0};\quad E(r) = \frac{E^0(\rho)}{E^0(b)};\quad u = \frac{u^0}{b}.$$

1. INTRODUCTION

Only a few papers have been devoted to the optimal design of nonhomogeneous annular discs. The dependence of stresses and displacements on the radius of rotating discs has been investigated by Pardoen *et al.* (1981) for power law distribution of Young's modulus, thickness and density of material. The aim of this investigation was to obtain the desirable stress state. The design of nonhomogeneous annular discs reinforced by equi-strained fibers has been considered by Bushmanov and Nemirovsky (1983a, b) as an example of the optimal design of nonhomogeneous reinforced plates. The design of continuous nonhomogeneous equi-strength annular discs of constant, Heinloo (1987a) and Hein *et al.* (1987) or variable, Hein and Heinloo (1987, 1988), Yhe Kai-Yuan and Liu Ping (1986), thickness has been examined. The method used in these papers and in the present work is based on the solution of the elastic problem with the additional requirement that the discs change from elastic to plastic state simultaneously at all points while the quasistatic loads achieve their limit values. The equi-strength state for rotating discs with regard to the Von Mises yield criterion was obtained by Yhe Kai-Yuan and Liu Ping (1986) by an appropriate choice of thickness distribution. In the rest of the above-mentioned papers the equi-strength state with regard to the Von Mises and Tresca yield criteria was obtained by choosing suitable Young's modulus distributions. Multilayer annular discs, made of a given number of concentric homogeneous rings joined together, were examined by Heinloo (1987b). In this investigation two piecewise constant Young's modulus distributions have been found. These guarantee, respectively, the global maximum and global minimum of the pressures upon the external contours with the condition that the Tresca yield criterion is satisfied on the inner radii of all layers. On the basis of the results given by Heinloo (1987a) it is not difficult to establish that if Poisson's ratios and yield stresses are the same as for the piecewise constant Young's modulus distribution the latter converges to the corresponding continuous distribution as the number of the layers is increased.

2. BASIC EQUATIONS

Let us assume that the annular discs are made of an isotropic, linearly elastic material which obeys the Tresca yield criterion. The Poisson's ratio and the yield stress are constants whereas the Young's modulus and the thickness are functions of the radial coordinate r.

For nonhomogeneous annular discs with variable thickness under quasistatic pressures upon the inner and external contours the stresses σ_r and σ_θ must satisfy the equilibrium equation

$$r\sigma_r' + (rs'+1)\sigma_r - \sigma_\theta = 0 \tag{1}$$

and the compatibility equation

$$r(\sigma_\theta' - \nu\sigma_r') - rq'(\sigma_\theta - \nu\sigma_r) + (1+\nu)(\sigma_\theta - \sigma_r) = 0, \tag{2}$$

where $s = \ln h(r)$ and $q = \ln E(r)$ (see, for example, Timoshenko and Goodier, 1970). In eqns (1), (2) and in all other formulae in this paper the primes denote differentiation with respect to r. We consider the limit state of the annular disc under uniformly distributed pressures. Therefore, the boundary conditions are

$$\sigma_r(\alpha) = -p_1 \tag{3}$$

$$\sigma_r(1) = -p_2. \tag{4}$$

We assume, that $p_1 + p_2 > 0$. The analysis is based on the Tresca yield criterion from which for the plane stress case we obtain the following restrictions:

$$|\sigma_\theta - \sigma_r| \leqslant 1 \tag{5}$$

$$|\sigma_\theta| \leqslant 1 \tag{6}$$

$$|\sigma_r| \leqslant 1. \tag{7}$$

These conditions establish that all possible stress states lie within or on the Tresca yield surface. If in one or two of the conditions (5)–(7) the sign of equality is valid for all radial coordinates r, the stress state lies on the yield surface and the annular disc is fully plastic.

3. STATICALLY ADMISSIBLE SOLUTIONS

First, let us assume that $\sigma_\theta = C = \text{const}$ for all radial coordinates r. On the basis of this assumption and making use of the boundary condition (4), the differential eqn (1) has the following solution

$$\sigma_r = -\frac{1}{rh(r)}\left[p_2 + C\int_r^1 h(r)\,\mathrm{d}r\right]. \tag{8}$$

According to the Tresca yield criterion the disc is fully plastic if $C = \pm 1$. Let us consider the case $C = -1$. Then from eqn (8) and the boundary condition (3), we get $p_1 = p_{1L}$, where

$$p_{1L} = \frac{1}{\alpha h(\alpha)}\left[p_2 - \int_\alpha^1 h(r)\,\mathrm{d}r\right]. \tag{9}$$

The parameter p_{1L}, defined by eqn (9), is the static limit inner pressure of the annular discs. It is easy to show that the conditions (5) and (7) are valid, if

$$-1 \leqslant \sigma_r \leqslant 0 \tag{10}$$

for $r \in [\alpha, 1]$. Taking into account the eqn (8), the conditions (10) can be rewritten as

$$\int_\alpha^1 h(r)\,\mathrm{d}r \leqslant p_2 \leqslant \min_r\left[rh(r) + \int_r^1 h(r)\,\mathrm{d}r\right] \tag{11}$$

which give the bounds to the limit external pressure.

For illustration we shall make use of the following special form for function $h(r)$: $h(r) = r^n$, where n $(-1 < n < 1)$ is a constant. Then eqn (9) and the inequalities (11) become

$$p_{1L} = \frac{1}{\alpha^{n+1}}\left[p_2 - \frac{1-\alpha^{n+1}}{n+1}\right]; \tag{12}$$

$$\frac{1-\alpha^{n+1}}{1+n} \leqslant p_2 \leqslant \begin{cases} \dfrac{1+n\alpha^{n+1}}{1+n}, & \text{when } n \geqslant 0 \\ 1, & \text{when } n \leqslant 0. \end{cases} \tag{13}$$

For the case of $C = 1$ we obtain $\sigma_r < 0$ from eqn (8). Hence, it is evident that the inequality (5) does not remain valid and, therefore, this problem does not have a statically admissible solution.

If $\sigma_\theta - \sigma_r = C_1 = \text{const}$ for all radial coordinates r, the solution of eqn (1) which satisfies the boundary condition (4), has the form

$$\sigma_r = -\frac{1}{h(r)}\left[p_2 + C_1\int_r^1 \frac{h(r)}{r}\,\mathrm{d}r\right]. \tag{14}$$

In order to satisfy the Tresca yield criterion for all radial coordinates r we must put $C_1 = 1$ or $C_1 = -1$. First let us take $C_1 = 1$. After fulfilling the boundary condition (5), it follows from (14) that $p_1 = p_{1u}$, where

$$p_{1u} = \frac{1}{h(\alpha)}\left[p_2 + \int_{\alpha}^{1} \frac{h(r)}{r}\,\mathrm{d}r\right]. \tag{15}$$

Eqn (15) gives the static limit inner pressure p_{1u} for the case $\sigma_\theta - \sigma_r = 1$. Employing the formula (14) and the equality $\sigma_\theta - \sigma_r = 1$ from inequalities (6) and (7) we get the following upper bound for the limit external pressure

$$p_2 \leqslant \min_r \left[h(r) - \int_r^1 \frac{h(r)}{r}\,\mathrm{d}r\right]. \tag{16}$$

Since $p_2 \geqslant 0$, we obtain from (16) the condition

$$\min_r \left[h(r) - \int_r^1 \frac{h(r)}{r}\,\mathrm{d}r\right] \geqslant 0 \tag{17}$$

that must be satisfied for the statically admissible solution to exist.

If the thickness distribution is given, for instance, in the form of the power function $h(r) = r^n$ for $-1 < n < 1$, we have from (15), (16) and (17) the following formulae:

$$p_{1u} = \frac{1}{\alpha^n}\left(p_2 + \frac{1-\alpha^n}{n}\right) \tag{18}$$

$$p_2 \leqslant \frac{(1+n)\alpha^n - 1}{n} \tag{19}$$

$$\alpha \geqslant n\sqrt{\frac{1}{n+1}}. \tag{20}$$

For $C_1 = -1$ this problem does not have a statically admissible solution. Indeed, rewriting eqn (14) as

$$\sigma_r = -\frac{1}{h(r)}\left[p_1 h(\alpha) - C_1 \int_{\alpha}^{r} \frac{h(r)}{r}\,\mathrm{d}r\right] \tag{21}$$

one can easily show that $\sigma_r < 0$ for $C_1 = -1$, thereby violating inequality (6).

In the case of homogeneous discs with $h(r) = r^n$, the stresses σ_r and σ_θ can be calculated according to the following formulae (see, for example, Timoshenko and Goodier, 1970):

$$\sigma_r = C_4 r^{x_1} + C_5 r^{x_2}; \quad \sigma_\theta = -C_4(1+x_2)r^{x_1} - C_5(1+x_1)r^{x_2} \tag{22}$$

where the constants C_4 and C_5 are determined from the boundary conditions (3), (4) and they have the following form:

$$C_4 = \frac{p_1 - p_2\alpha^{x_2}}{\alpha^{x_2} - \alpha^{x_1}}, \quad C_5 = \frac{p_1 - p_2\alpha^{x_1}}{\alpha^{x_1} - \alpha^{x_2}}. \tag{23}$$

Here x_1 and x_2 are the solutions of the quadratic

$$x^2 + (2+n)x + (1+\nu)n = 0.$$

It is not difficult to establish that $0 \leqslant x_1 \leqslant -n$, $-2 \leqslant x_2 \leqslant -1$ for $n \leqslant 0$ and $-n < x_1 < 0$, $-3 < x_2 \leqslant -2$ for $n \geqslant 0$.

Substituting eqns (22) into conditions (5)–(7) and taking into account eqns (23), we obtain

$$|(p_1-p_2\alpha^{x_2})r^{x_1}-(p_1-p_2\alpha^{x_1})r^{x_2}| \leqslant \alpha^{x_2}-\alpha^{x_1}$$
$$|(p_1-p_2\alpha^{x_2})(2+x_2)r^{x_1}-(p_1-p_2\alpha^{x_1})(2+x_1)r^{x_2}| \leqslant \alpha^{x_2}-\alpha^{x_1}$$
$$|(p_1-p_2\alpha^{x_2})(1+x_2)r^{x_1}-(p_1-p_2\alpha^{x_1})(1+x_1)r^{x_2}| \leqslant \alpha^{x_2}-\alpha^{x_1}. \tag{24}$$

Functions which lie within the modulus signs are monotonic functions for $n = 0$, whereas $x_1 = 0$ and $x_2 = -2$. Let us assume that n is so small that the monotonicity property will be preserved. Under this assumption the inequalities (24) are satisfied, if $p_1 \leqslant 1$; $p_2 \leqslant 1$

$$|[(2+x_1)\alpha^{x_2}-(2+x_2)\alpha^{x_1}]p_1-\alpha^{x_1}\alpha^{x_2}(x_1-x_2)p_2| \leqslant \alpha^{x_2}-\alpha^{x_1}$$
$$|[(2+x_1)\alpha^{x_1}-(2+x_2)\alpha^{x_2}]p_2+(x_2-x_1)p_1| \leqslant \alpha^{x_2}-\alpha^{x_1}$$
$$|[(1+x_1)\alpha^{x_2}-(1+x_2)\alpha^{x_1}]p_1-\alpha^{x_1}\alpha^{x_2}(x_1-x_2)p_2| \leqslant \alpha^{x_2}-\alpha^{x_1}$$
$$|[(1+x_1)\alpha^{x_1}-(1+x_2)\alpha^{x_2}]p_2+(x_2-x_1)p_1| \leqslant \alpha^{x_2}-\alpha^{x_1}. \tag{25}$$

Solving the system (25) with respect to p_1, we get

$$\begin{aligned} 0 \leqslant p_1 \leqslant p^0_{1u}, &\quad \text{for } 0 \leqslant p_2 \leqslant p^0_{2L} \\ p^0_{1L} \leqslant p_1 \leqslant p^0_{1u}, &\quad \text{for } p^0_{2L} \leqslant p_2 \leqslant p^0_{2u} \end{aligned} \tag{26}$$

where

$$p^0_{2L} = \frac{\alpha^{x_2}-\alpha^{x_1}}{\alpha^{x_1}\alpha^{x_2}(x_1-x_2)}$$

$$p^0_{2u} = \begin{cases} \dfrac{(2+x_1)\alpha^{x_2}-(2+x_2)\alpha^{x_1}}{\alpha^{x_1}\alpha^{x_2}(x_1-x_2)}, & \text{when } n \geqslant 0 \\ 1, & \text{when } n \leqslant 0. \end{cases}$$

Here, p^0_{1L} and p^0_{1u} are calculated from

$$p^0_{1L} = \frac{\alpha^{x_1}-\alpha^{x_2}+\alpha^{x_1}\alpha^{x_2}(x_1-x_2)p_2}{(1+x_1)\alpha^{x_2}-(1+x_2)\alpha^{x_1}}, \quad \text{when } p^0_{2L} \leqslant p_2 \leqslant p^0_{2u} \tag{27}$$

and

$$p^0_{1u} = \frac{\alpha^{x_2}-\alpha^{x_1}+\alpha^{x_1}\alpha^{x_2}(x_1-x_2)p_2}{(2+x_1)\alpha^{x_2}-(2+x_2)\alpha^{x_1}}, \quad \text{when } 0 \leqslant p_2 \leqslant p^0_{2k} \tag{28}$$

where $p^0_{2k} = \min(1, p^0_{2M})$, with

$$p^0_{2M} = \frac{(1+x_1)\alpha^{x_2}-(1+x_2)\alpha^{x_1}}{\alpha^{x_1}\alpha^{x_2}(x_1-x_2)}.$$

For $p^0_{2k} \leqslant p_2 \leqslant p^0_{2u}$, we have $p^0_{1u} = 1$.

It has been verified numerically that (26) is the solution of the system (24) for $-1.0 < n < 1.0$; $0.1 < \alpha < 1.0$; $0.2 < \nu < 0.35$.

Constraints (26) determine the region to the elastic deformations for homogeneous annular discs.

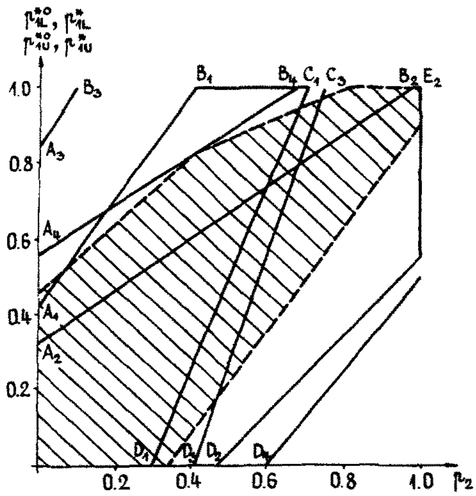

Fig. 1. The dependences of the limit pressures and regions of the elastic deformations.

The variations of p_{1u}, p_{1L}, p^0_{1u}, p^0_{1L} on p_2, calculated by the formulae (12) (C_3D_3 and C_4D_4), (18) (A_3B_3 and A_4B_4), (27) (C_1D_1 and C_2D_2), (28) (A_1B_1 and A_2B_2) with $p^0_{1u} = p^0_{2u} = 1$ (B_1C_1, B_2E_2 and E_2C_2) for $\nu = 0.33$, $\alpha = 0.50$, $n = -0.60$ (C_4D_4, A_4B_4, C_2D_2 and A_2B_2), $n = 0.60$ (C_3D_3, A_3B_3, C_1D_1 and A_1B_1) are shown in Fig. 1. The regions of the elastic deformations for homogeneous discs are located within the polygon $A_1B_1C_1D_1$ for $n = 0.60$ and the polygon $A_2B_2E_2C_2D_2$ for $n = -0.60$. The elastic region of a non-homogeneous disc with the optimal distribution of the Young's modulus has been shaded in Fig. 1 for $n = -0.60$, whereas the Tresca yield criterion is satisfied simultaneously at all points for $p_{1u} = 0.83$, $p_2 = 0.40$. The algorithm for the calculation of this region is described in Section 8.

The efficiency of the optimal design may be estimated by means of the ratios p^0_{1u}/p_{1u} or p^0_{1L}/p_{1L}. For example, it follows from Fig. 1 that $p^0_{1u}/p_{1u} = 0.50$ for $n = 0.60$ and $p^0_{1u}/p_{1u} = 0.58$ for $n = -0.60$, when $p_2 = 0$. Therefore, the limit inner pressure p^0_{1u} of homogeneous discs is nearly a half of the limit pressure p_{1u} for discs of optimal non-homogeneity in the region of the elastic deformations.

4. OPTIMAL DISTRIBUTIONS OF YOUNG'S MODULUS

The stress components must satisfy the compatibility eqn (2). The compatibility eqn (2) which must be satisfied by the stresses will be used to find the distribution of Young's modulus corresponding to a prescribed stress state. This distribution guarantees the satisfaction of Tresca yield criterion simultaneously at all points of the disc while the pressures achieve their limit values found above.

Let us consider the case when the statically admissible solution is determined by (8) for $C = -1$ and $\sigma_\theta = -1$. Then eqn (2) can be expressed in the form

$$q' = \frac{(1 - \nu r s')\sigma_r + 1}{r(1 + \nu\sigma_r)}. \tag{29}$$

For given distribution of thickness the function q can be found by integrating eqn (29) with initial condition $q(1) = 0$. The distribution of Young's modulus is calculated from

$$E(r) = \exp q(r). \tag{30}$$

If the distribution of thickness is the power law $h(r) = r^n$, eqn (29) becomes

$$q' = \frac{(1-\nu n)\sigma_r + 1}{r(1+\nu\sigma_r)} \tag{31}$$

where

$$\sigma_r = \frac{1}{r^{n+1}}\left(\frac{1-r^{n+1}}{1+n} - p_2\right). \tag{32}$$

For constant thickness we have $n = 0$ and eqn (31) takes the form

$$q' = \frac{1-p_2}{r[r(1-\nu)+\nu(1-p_2)]}$$

which gives

$$E(r) = \left[\frac{r(1-\nu p_2)}{r(1-\nu)+\nu(1-p_2)}\right]^{1/\nu}. \tag{33}$$

Let us assume that the statically admissible solution (14) for $C_1 = 1$ and $\sigma_\theta - \sigma_r = 1$ is given. Now, instead of eqn (29) we have

$$q' = \frac{2-(1-\nu)rs'\sigma_r}{r[1+(1-\nu)\sigma_r]}. \tag{34}$$

Using the function $h(r) = r^n$ in eqn (34), we obtain

$$q' = \frac{2-(1-\nu)n\sigma_r}{r[1+(1-\nu)\sigma_r]} \tag{35}$$

where

$$\sigma_r = -\frac{1}{r^n}\left(p_2 + \frac{1-r^n}{n}\right). \tag{36}$$

For the special case $n \to 0$ eqn (35) will become

$$q' = \frac{2}{r[(1-\nu)(\ln r - p_2)+1]}$$

which has the following exact solution:

$$E(r) = \left[\frac{(1-\nu)(\ln r - p_2)+1}{1-(1-\nu)p_2}\right]^{2/(1-\nu)}. \tag{37}$$

Equations (32) and (36) determine the distribution of the radial stress at the instant when the limit state is reached.

The variation of Young's modulus with r, calculated from eqn (30), after the numerical integration of the differential eqn (35) (Curves 1 and 2) and of (31) (Curves 3–5) using the

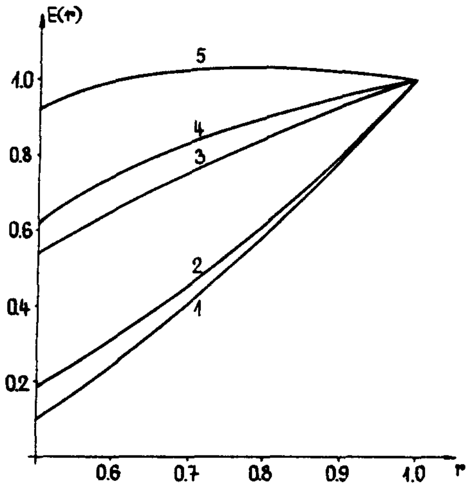

Fig. 2. The optimal distributions of Young's modulus.

standard 4th order Runge–Kutta method for $q(1) = 0$ is shown in Fig. 2. Curve 1 corresponds to the case $n = 0.60$, $p_{1u} = 0.85$, $p_2 = 0.00$; Curve 2—$n = -0.60$, $p_{1u} = 0.57$, $p_2 = 0.00$; Curve 3—$n = 0.60$; $p_{1L} = 0.00$, $p_2 = 0.42$; Curve 4—$n = -0.60$, $p_{1L} = 0.00$, $p_2 = 0.60$ and Curve 5—$n = -0.60$, $p_{1L} = 0.50$, $p_2 = 0.98$. The results based on eqns (33) and (37) have been presented previously by Heinloo (1987a).

5. KINEMATICALLY ADMISSIBLE SOLUTIONS

Up to now we considered the case of a given distribution of thickness. Now, let us determine the optimal distribution of thickness for a given nonhomogeneity.

Let us assume that $\sigma_\theta = C = \text{const}$ for all radial coordinates r as in the previous Sections. Now, integrating eqn (2) and taking into account the boundary condition (4), we get

$$\sigma_r = \frac{C}{v} + E(r)r^{-(1+v)/v}\left[\frac{1-v^2}{v^2}C\int_r^1 \frac{r^{1/v}}{E(r)}\mathrm{d}r - \frac{C}{v} - p_2\right]. \tag{38}$$

To satisfy the Tresca yield criterion, we must take $C = -1$. Now, from (38) and the boundary condition (3) we obtain $p_1 = p_{1L}$ in the limit state. The limit inner pressure p_{1L} is defined by the formula

$$p_{1L} = \frac{1}{v} + E(\alpha)\alpha^{-(1+v)/v}\left[\frac{1-v^2}{v^2}\int_\alpha^1 \frac{r^{1,v}}{E(r)}\mathrm{d}r - \frac{1}{v} + p_2\right]. \tag{39}$$

In this case the bounds of limit external pressure, analogous to the restrictions (11), are

$$\min_r\left[g(r) + \frac{r^{(1+v)/v}}{E(r)}\right] \geqslant p_2 \geqslant \max_r g(r) \tag{40}$$

where

$$g(r) = \frac{1}{v} - \frac{1-v^2}{v^2}\int_r^1 \frac{r^{1/v}}{E(r)}\mathrm{d}r - \frac{r^{(1+v)/v}}{E(r)}.$$

on the special case when Young's modulus is the power function $E(r) = r^n$, eqn (39) and the inequality (40) becomes

$$p_{1L} = p_2\alpha^k - \frac{(\alpha^k - 1)(n-\nu-1)}{n\nu-\nu-1} \tag{41}$$

and

$$\begin{aligned} \alpha^{-k} + M \geqslant p_2 \geqslant 0, &\quad \text{when } 1+\nu \leqslant n \\ \alpha^{-k} + M \geqslant p_2 \geqslant M, &\quad \text{when } 0 \leqslant n \leqslant 1+\nu \\ 1 \geqslant p_2 \geqslant M, &\quad \text{when } n \leqslant 0 \end{aligned}$$

where

$$M = \frac{(\alpha^k - 1)(n-\nu-1)}{\alpha^k(n\nu-\nu-1)}; \quad k = (n\nu-\nu-1)\bar{\nu}^1.$$

The case $C = 1$ is not possible because the inequality (5) is not satisfied, at least on the contour of the disc.

Finally, let us consider the case where $\sigma_\theta - \sigma_r = C_1 = \text{const}$ for all radial coordinates r. Using the boundary condition (4), we can solve the differential equation (2) to get

$$\sigma_r = E(r)\left[\frac{C_1(1+\nu)}{1-\nu}\int_r^1 \frac{dr}{rE(r)} - p_2\right] - \frac{[1-E(r)]C_1}{1-\nu}. \tag{42}$$

To satisfy the Tresca yield criterion, we must take $C_1 = 1$. Now, from (42) and (3), we get $p_1 = p_{1u}$ in the limit state, where

$$p_{1u} = E(\alpha)\left[p_2 - \frac{1+\nu}{1-\nu}\int_\alpha^1 \frac{dr}{rE(r)}\right] + \frac{1-E(\alpha)}{1-\nu}. \tag{43}$$

Instead of restrictions (40) we obtain

$$\min_r\left[f(r) + \frac{1}{E(r)}\right] \geqslant p_2 \geqslant \max_r f(r) \tag{44}$$

where

$$f(r) = \frac{1}{1-\nu}\left[(1+\nu)\int_r^1 \frac{dr}{rE(r)} - \frac{1-E(r)}{E(r)}\right].$$

Let $E(r) = r^n$. In this particular case eqn (43) and the inequality (44) become

$$p_{1u} = p_2\alpha^n - \frac{(1-\alpha^n)(1+\nu-n)}{n(1-\nu)} \tag{45}$$

and

$$\begin{aligned} 1 \geqslant p_2 \geqslant N, &\quad \text{when } 1+\nu \geqslant n \\ 1 \geqslant p_2 \geqslant 0, &\quad \text{when } 1+\nu \leqslant n \leqslant (1+\nu)\nu^{-1} \\ \alpha^{-n} + N \geqslant p_2 \geqslant 0, &\quad \text{when } (1+\nu)\nu^{-1} \leqslant n \end{aligned}$$

where

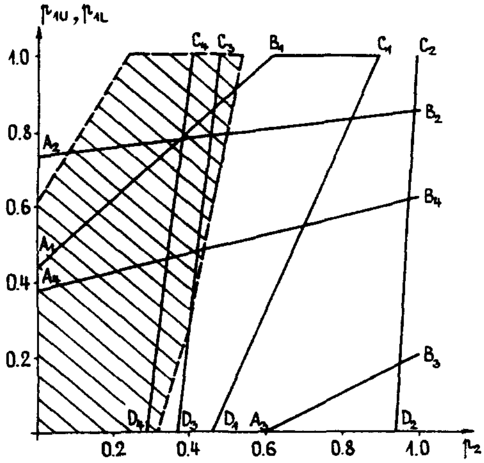

Fig. 3. The dependences of the limit pressures and regions of the elastic deformations.

$$N = \frac{(\alpha^{-n}-1)(1+\nu-n)}{n(1-\nu)}.$$

The case $C_1 = -1$ is not possible because the inequality (6) is not satisfied, at least in the neighbourhood of the contour.

The relation between p_{1L}, p_{1u} and p_2 based on eqn (41) (C_2D_2, C_3D_3 and C_4D_4), eqn (45) (A_2B_2, A_3B_3 and A_4B_4) for $\nu = 0.33$, $\alpha = 0.50$, $n = 0.00$ (C_2D_2), $n = 0.90$ (A_3B_3 and C_3D_3), $n = 3.00$ (A_2B_2), $n = 1.00$ (C_4D_4), $n = 2.00$ (A_4B_4) is shown in Fig. 3. The region of elastic deformations for a nonhomogeneous disc with constant thickness lies inside the polygon $A_1B_1C_1D_1$ for $n = 0.9$. The analogous region, in case of the disc with the optimal distribution of thickness, lies inside the shaded polygon whereas the Tresca yield criterion is satisfied simultaneously at all points of the disc for the pressures $p_{1L} = 0.27$ and $p_2 = 0.40$. The algorithm for the calculation this region is described in Section 8.

6. OPTIMAL DISTRIBUTIONS OF THICKNESS

Now from the equilibrium eqn (1) we may find the thickness distribution corresponding to the prescribed stress state. This distribution guarantees the satisfaction of the Tresca yield criterion simultaneously at all points of the disc, when the pressures achieve their limit values found in Section 5.

Let us assume that the kinematically admissible solution is given by eqn (38) for $C = -1$ and $\sigma_\theta = -1$. Then, from (1) we obtain the following differential equation for s:

$$s' = \frac{-1-\sigma_r - r\sigma_r'}{r\sigma_r}. \tag{46}$$

Integrating (46) for the given distribution of Young's modulus $E(r)$ and taking into account the initial condition $s(1) = 0$, we obtain the function $s(r)$. Then the optimal distribution of thickness can be found from

$$h(r) = \exp s(r). \tag{47}$$

Let us consider the particular case when the nonhomogeneity is defined by the function $E(r) = r^n$. It follows from (38) that the radial stress has the form

$$\sigma_r = \frac{(r^k - 1)(n-v-1)}{nv-v-1} - p_2 r^k; \quad k = (nv-v-1)\bar{v}^{1}. \tag{48}$$

For a constant Young's modulus i.e. for $n = 0$ we get

$$\sigma_r = -1 + (1-p_2) r^{-(1+v)/v} \tag{49}$$

and (46) gives the following exact solution

$$h(r) = \left[\frac{p_2}{1-(1-p_2)r^{-m}}\right]^{1/(1+v)}$$

where

$$m = (1+v)v^{-1}.$$

Let us assume that the kinematically admissible solution is determined by eqn (42) for $C_1 = 1$ and $\sigma_\theta - \sigma_r = 1$.

From eqn (1) we obtain instead of (46)

$$s' = \frac{1 - r\sigma_r'}{r\sigma_r}. \tag{50}$$

For $E(r) = r^n$ the radial stress component σ_r in eqn (50) can be calculated by the formula

$$\sigma_r = \frac{(1-r^n)(1+v-n)}{n(1-v)} - p_2 r^n. \tag{51}$$

For the special case $n \to 0$, we obtain from equality (51)

$$\sigma_r = -p_2 - \frac{1+v}{1-v}\ln r, \tag{52}$$

and eqn (50) has an exact solution

$$h(r) = \left[\frac{p_2}{p_2 + g\ln r}\right]^{2/(1+v)}$$

where

$$g = \frac{1+v}{1-v}.$$

Notice that (48), (49), (51), (52) determine the distributions of radial stress σ_r at the moment when the limit state is reached.

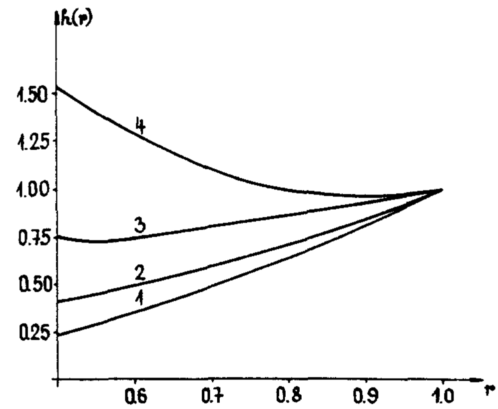

Fig. 4. The optimal distribution of thickness.

Figure 4 presents the results obtained from (47) after numerically integrating the differential equation (46) (Curves 1–3) and (50) (Curve 4) with the condition $s(1) = 0$, using the standard 4th ordered Runge–Kutta method. Curve 1 corresponds to $n = 1.00$, $p_{1L} = 0.90$, $p_2 = 0.40$; Curve 2—$n = 0.90$, $p_{1L} = 0.27$, $p_2 = 0.40$; Curve 3—$n = 0.70$, $p_{1L} = 0.34$, $p_2 = 0.55$; Curve 4—$n = 3.00$, $p_{1u} = 0.78$, $p_2 = 0.40$.

Besides the nonhomogeneous annular disc with optimal distribution of thickness, we consider a disc with constant thickness h_o. Assuming that these discs have the same volume, we find that h_o is given by

$$h_o = \frac{2}{1-\alpha^2}\int_\alpha^1 rh(r)\,dr.$$

The limit forces upon the inner and external contours of the discs under consideration may be calculated from $T_1 = 2\pi\alpha h(\alpha)p_1$, $T_1^0 = 2\pi\alpha h_o p_1^0$, $T_2 = 2\pi p_2$, $T_2^0 = 2\pi h_o p_2^0$, where p_1^0 and p_2^0 stand for the limit internal and external pressures for discs of constant thickness in the region of the elastic deformations. The economy of the nonhomogeneous annular discs with the optimal distribution of thickness compared to discs of constant thickness may be assessed by the ratios T_1/T_1^0 for $T_2 = T_2^0$ or T_2/T_2^0 for $T_1 = T_1^0$. For instance, we have $T_2/T_2^0 = 1.05$ for $n = 0.90$, $p_{1L} = 0.27$, $p_2 = 0.40$ and $T_1/T_1^0 = 7.57$ for $n = 3.0$, $p_{1u} = 0.78$, $p_2 = 0.40$.

7. RECIPROCAL METHOD

In the previous Sections it was assumed that the distribution of either Young's modulus or thickness was given. If neither is prescribed a reciprocal method may be used. Let us consider the case where the distribution of the radial stress σ_r is given by the linear function

$$\sigma_r = Ar + B. \tag{53}$$

From boundary conditions (3) and (4), it follows that

$$A = \frac{p_1 - p_2}{1-\alpha}; \quad B = \frac{p_2\alpha - p_1}{1-\alpha}.$$

To satisfy the Tresca yield criterion simultaneously at all points, we must take $\sigma_\theta = -1$ for all values of r. The inequalities (5) and (6) are satisfied, if

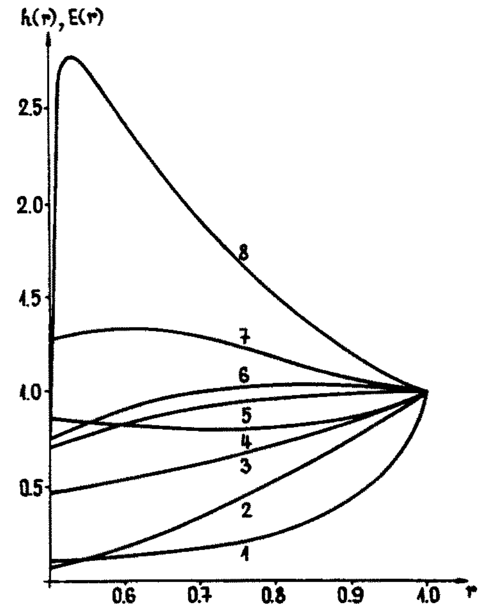

Fig. 5. The optimal distributions of Young's modulus and thickness.

$$0 \leqslant p_1 \leqslant 1; \quad 0 \leqslant p_2 \leqslant 1. \tag{54}$$

The optimal distributions of thickness and Young's modulus are found from the equilibrium (1) and the compatibility equations (2), respectively, and are given by

$$h(r) = \left|\frac{Ar+B}{A+B}\right|^{(1-B)/B} r^{-(1+B)/B} \tag{55}$$

$$E(r) = \left|\frac{Avr+Bv+1}{Av+Bv+1}\right|^{[1+v+v^2(B-1)]/[v(1+vB)]} r^{[(1+B)(1+v)]/[1+vB]}. \tag{56}$$

If instead of the equality $\sigma_\theta = -1$, if we take $\sigma_\theta - \sigma_r = 1$, the condition (54) is preserved, but the functions $h(r)$ and $E(r)$ are given by

$$h(r) = \left|\frac{A+B}{Ar+B}\right|^{(1-B)/B} r^{1/B} \tag{57}$$

$$E(r) = \left|\frac{rA(1-v)+B(1-v)+1}{A(1-v)+B(1-v)+1}\right|^{[B-v(1+B)]/[1+B(1-v)]} r^{[1+v]/[1+B(1-v)]}. \tag{58}$$

The distributions of $E(r)$ (Curves 2, 3, 5, 6) and $h(r)$ (Curves 1, 4, 7, 8), calculated from (55) (Curves 1, 7, 8), (56) (Curves 3, 5, 6), (57) (Curve 4) and (58) (Curve 2) for $v = 0.33$, $\alpha = 0.5$ are presented in Fig. 5. Curves 1–4 correspond to the case $p_1 = 0.90$, $p_2 = 0.20$, Curves 5, 7—$p_1 = 0.30$, $p_2 = 0.80$ and Curves 6, 8—$p_1 = 0.00$, $p_2 = 0.90$.

8. REGIONS OF ELASTIC DEFORMATIONS

Using the equilibrium eqn (1), the compatibility equation of strains (2) may be rewritten as

$$\sigma_r''+f_1(r)\sigma_r'+f_2(r)\sigma_r = 0 \tag{59}$$

where

$$f_1(r) = s'-q'+\frac{3}{r}$$

$$f_2(r) = \frac{1}{r}[s''r+s'(2+\nu)-q'(1-\nu+s'r)]$$

and

$$s = \ln h(r),\quad q(r) = \ln E(r).$$

In order to determine the ranges of pressures p_1 and p_2 within which the deformations of the disc are elastic, eqn (59) was used. First we determined the radial stress σ_r by solving the boundary value problem (59), (3), (4) and then found the hoop stress component from the equilibrium eqn (1). Finally by incrementing the parameters p_1 and p_2 stepwise we obtained the desired region in the p_1-p_2 plane. In this region the stresses σ_r and σ_θ satisfy the conditions (5)–(7). The boundary value problem (59), (3), (4) was reduced to two initial-value problems which were solved by the standard 4th order Runge–Kutta method. For the calculation of the derivative σ_r' in eqn (1) the 4th order formulae of the numerical differentiation were used. The results of these calculations were presented in Sections 3, 5.

9. CONCLUSION

The limit analysis of variable-thickness annular discs is important for engineers and designers in order to estimate their load carrying capacity. In the present paper, the optimal variations of Young's modulus or thickness with radius were found. These optimal distributions guarantee the satisfaction of the Tresca yield criterion simultaneously at all points under determined limit pressures. The optimal distributions of Young's modulus in multilayer discs made of a great number of concentric homogeneous rings of different Young's moduli may be obtained approximately. The examples given in the paper show the evident advantages of equi-strength discs over discs in which the equi-strength condition is not satisfied.

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